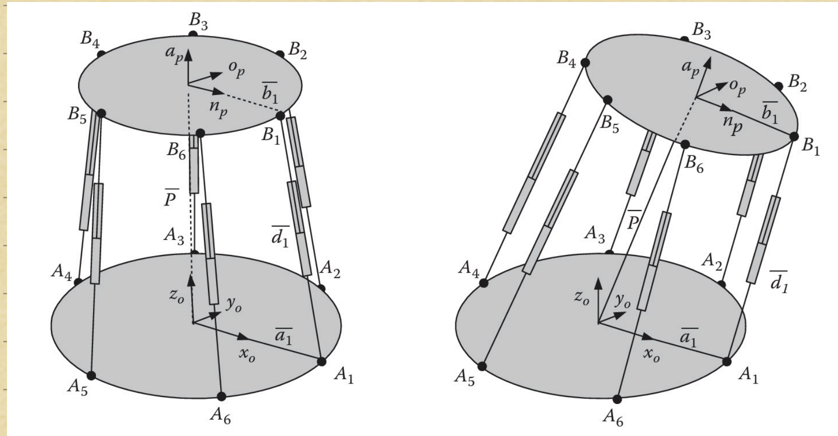
Inverse Kinematics



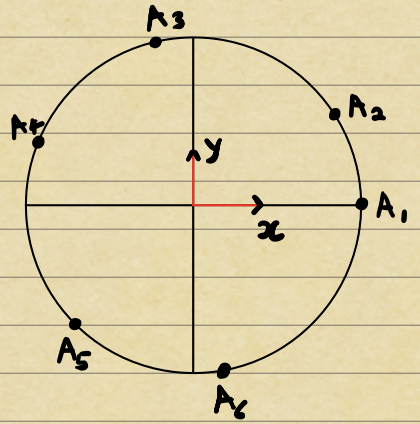
- $\bar{d}$ is the length of each leg
- $\bar{P}$ is the position vector of the platform

- Displacement vector: In order to move from the base frame x, y, z to the platform frame n_p, O_p, a_p the following vector sum is required.

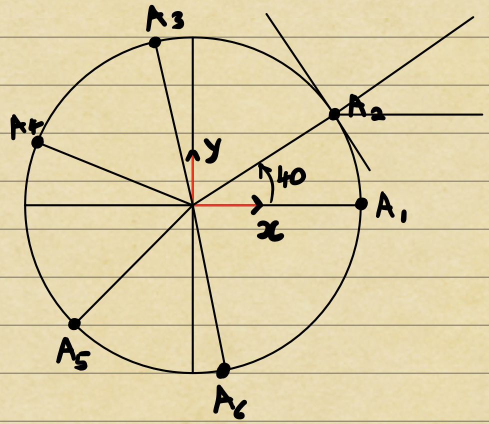
$$\bar{a} + \bar{d} - \bar{b} = \bar{P} \longrightarrow \bar{d} = \bar{P} + \bar{b} - \bar{a}$$

$$\bar{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} \quad \bar{a} = \begin{bmatrix} x_{oa,i} \\ y_{oa,i} \\ z_{oa,i} \\ 1 \end{bmatrix} \quad \bar{b} = R_x R_y R_z \begin{bmatrix} x_{b,i} \\ y_{b,i} \\ z_{b,i} \\ 1 \end{bmatrix}$$

Using a $0°-40°$ arrangement:

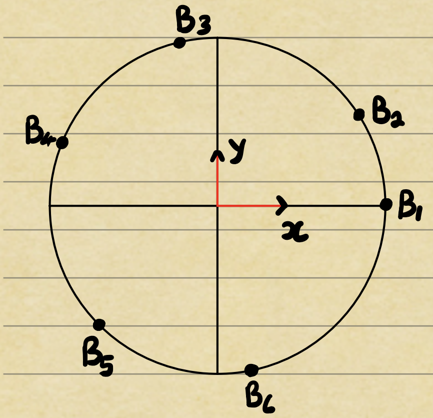


$A_1 \longrightarrow 0°$
$A_2 \longrightarrow 40°$
$A_3 \longrightarrow 120°$
$A_4 \longrightarrow 160°$
$A_5 \longrightarrow 240°$
$A_6 \longrightarrow 280°$

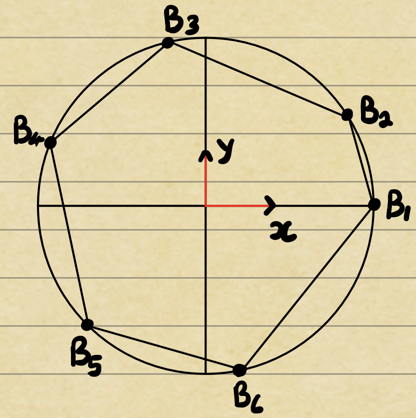


$a_x = a\cos(A) \quad a_y = a\sin(A)$

$a_{1x} = a\cos(0) \quad a_{1y} = a\sin(0)$
$a_{2x} = a\cos(40) \quad a_{2y} = a\sin(40)$
$a_{3x} = a\cos(120) \quad a_{3y} = a\sin(120)$
$a_{4x} = a\cos(160) \quad a_{4y} = a\sin(160)$
$a_{5x} = a\cos(240) \quad a_{5y} = a\sin(240)$
$a_{6x} = a\cos(280) \quad a_{6y} = a\sin(280)$



$B_1 \rightarrow 0°$
$B_2 \rightarrow 40°$
$B_3 \rightarrow 120°$
$B_4 \rightarrow 160°$
$B_5 \rightarrow 240°$
$B_6 \rightarrow 280°$

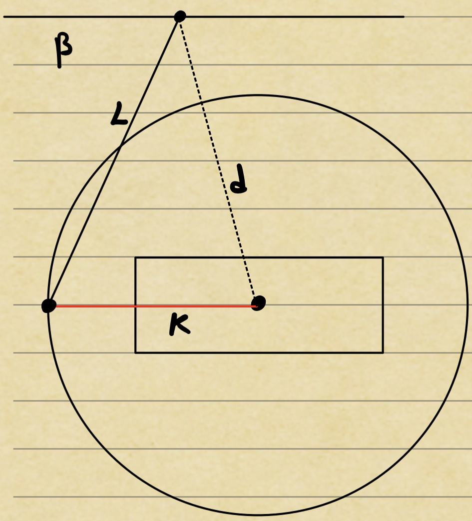


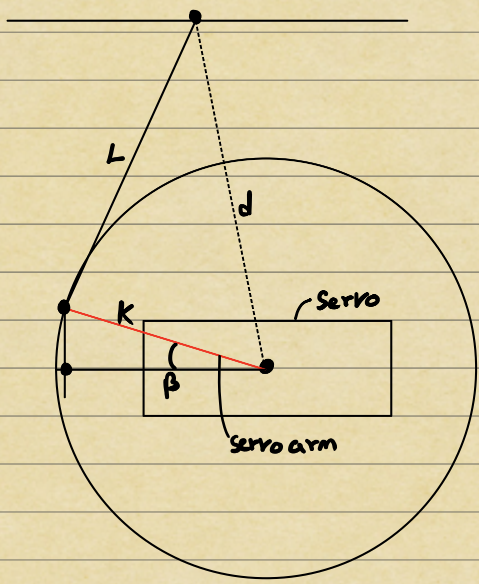
$a_x = a\cos(A) \quad a_y = a\sin(A)$

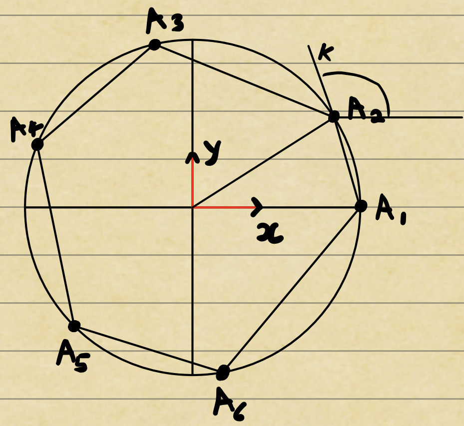
$b_{1x} = b\cos(0) \quad b_{1y} = b\sin(0)$
$b_{2x} = b\cos(40) \quad b_{2y} = b\sin(40)$
$b_{3x} = b\cos(120) \quad b_{3y} = b\sin(120)$
$b_{4x} = b\cos(160) \quad b_{4y} = b\sin(160)$
$b_{5x} = b\cos(240) \quad b_{5y} = b\sin(240)$
$b_{6x} = b\cos(280) \quad b_{6y} = b\sin(280)$

$$\begin{bmatrix} d_{ix} \\ d_{iy} \\ d_{iz} \\ 1 \end{bmatrix} = \overbrace{\begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} + R_z R_y R_x \begin{bmatrix} b_{ix} \\ b_{iy} \\ b_{iz} \\ 1 \end{bmatrix}}^{\vec{ob}} - \begin{bmatrix} a_{ix} \\ a_{iy} \\ a_{iz} \\ 1 \end{bmatrix}$$

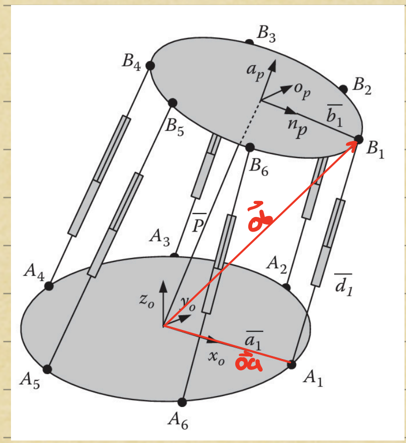
$$\boxed{|d_i| = \sqrt{d_{ix}^2 + d_{iy}^2 + d_{iz}^2}}$$

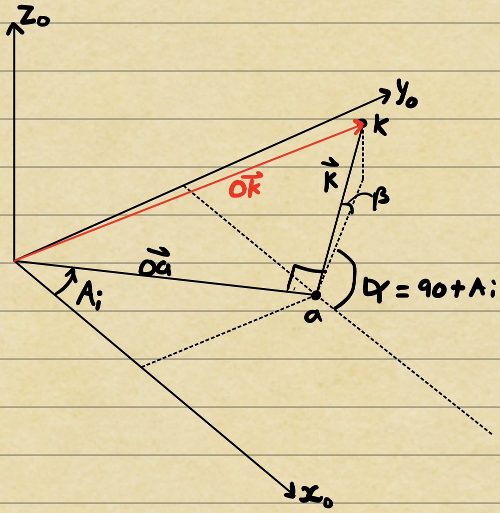






- K: Servo arm Length
- L: Length of rod connecting Servo to the platform
- d: Leg Length calculated from the inverse Kinematics
- β: Servo Rotation Angle



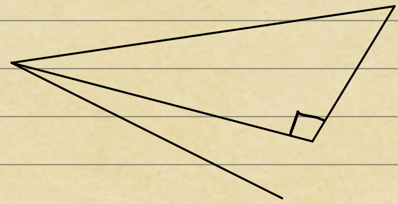


Calculating Servo Angle (β)

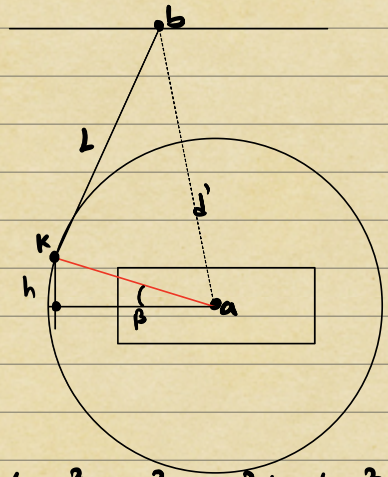
$$x_{ok} = k\cos(\beta)\cos(\alpha) + x_{oa} \quad ①$$

$$y_{ok} = k\cos(\beta)\sin(\alpha) + y_{oa} \quad ②$$

$$z_{ok} = k\sin(\beta) + z_{oa} \quad ③$$



$$|\vec{d}| = d^2 = |\vec{Ob}| - |\vec{Oa}| \qquad |\vec{ak}| = k^2 = |\vec{Oa}| - |\vec{Ok}|$$



$$\vec{Ob} = \vec{P} + R\vec{b}$$
$$\vec{Oa} = \vec{oa}$$

$$L^2 = |\vec{kb}| = |\vec{Ob}| - |\vec{Ok}|$$

$$\therefore \left.\begin{aligned} d^2 &= (x_{ob} - x_{oa})^2 + (y_{ob} - y_{oa})^2 + (z_{ob} - z_{oa})^2 \\ k^2 &= (x_{oa} - x_{ok})^2 + (y_{oa} - y_{ok})^2 + (z_{oa} - z_{ok})^2 \\ L^2 &= (x_{ob} - x_{ok})^2 + (y_{ob} - y_{ok})^2 + (z_{ob} - z_{ok})^2 \end{aligned}\right\}$$

$$(x_{oa}^2 + y_{oa}^2 + z_{oa}^2) + (x_{ob}^2 + y_{ob}^2 + z_{ob}^2) - 2(x_{oa}x_{ob} + y_{oa}y_{ob} + z_{oa}z_{ob}) = d^2 \quad ④$$

$$(x_{oa}^2 + y_{oa}^2 + z_{oa}^2) + (x_{ok}^2 + y_{ok}^2 + z_{ok}^2) - 2(x_{oa}x_{ok} + y_{oa}y_{ok} + z_{oa}z_{ok}) = k^2 \quad ⑤$$

$$(x_{ob}^2 + y_{ob}^2 + z_{ob}^2) + (x_{ok}^2 + y_{ok}^2 + z_{ok}^2) - 2(x_{ob}x_{ok} + y_{ob}y_{ok} + z_{ob}z_{ok}) = L^2 \quad ⑥$$

④ - ⑤ - ⑥ :

$$d^2 - k^2 - L^2 = -2(x_{ok}^2 + y_{ok}^2 + z_{ok}^2) - 2(x_{oa}x_{ob} + y_{oa}y_{ob} + z_{oa}z_{ob}) + 2(x_{oa}x_{ok} + y_{oa}y_{ok} + z_{oa}z_{ok}) + 2(x_{ob}x_{ok} + y_{ob}y_{ok} + z_{ob}z_{ok})$$

$$\boxed{d^2 - k^2 - L^2} = -2(x_{ok}^2 + y_{ok}^2 + z_{ok}^2) + 2x_{oa}(x_{ok} - x_{ob}) + 2y_{oa}(y_{ok} - y_{ob}) + 2z_{oa}(z_{ok} - z_{ob}) + 2(x_{ob}x_{ok} + y_{ob}y_{ok} + z_{ob}z_{ok})$$

$$= -2(x_{ok}^2 + y_{ok}^2 + z_{ok}^2) + 2x_{oa}(k\cos(\beta)\cos(\alpha) + x_{oa} - x_{ob}) + 2y_{oa}(k\cos(\beta)\sin(\alpha) + y_{oa} - y_{ob}) + 2z_{oa}(k\sin(\beta) + z_{oa} - z_{ob}) + 2x_{ob}(k\cos(\beta)\cos(\alpha) + x_{oa}) + 2y_{ob}(k\cos(\beta)\sin(\alpha) + y_{oa}) + 2z_{ob}(k\sin(\beta) + z_{oa})$$

$$= -2(x_{ok}^2 + y_{ok}^2 + z_{ok}^2) + 2x_{oa}(k\cos(\beta)\cos(\alpha) + x_{oa}) + 2y_{oa}(k\cos(\beta)\sin(\alpha) + y_{oa}) + 2z_{oa}(k\sin(\beta) + z_{oa}) \cancel{- 2x_{oa}x_{ob}} \cancel{- 2y_{oa}y_{ob}} \cancel{- 2z_{oa}z_{ob}} \cancel{+ 2x_{oa}x_{ob}} \cancel{+ 2y_{oa}y_{ob}} \cancel{+ 2z_{oa}z_{ob}} + 2x_{ob}k\cos(\beta)\cos(\alpha) + 2y_{ob}k\cos(\beta)\sin(\alpha) + 2z_{ob}k\sin(\beta)$$

$$'' = -2(x_{ok}^2 + y_{ok}^2 + z_{ok}^2) + 2x_{oa}(k\cos(\beta)\cos(\alpha) + x_{oa}) + 2y_{oa}(k\cos(\beta)\sin(\alpha) + y_{oa}) + 2z_{oa}(k\sin(\beta) + z_{oa})$$
$$+ 2x_{ob}k\cos(\beta)\cos(\alpha) + 2y_{ob}k\cos(\beta)\sin(\alpha) + 2z_{ob}k\sin(\beta)$$

$$'' = \overbrace{-2(x_{ok}^2 + y_{ok}^2 + z_{ok}^2)}^{\text{simplify}} + 2(x_{oa}^2 + y_{oa}^2 + z_{oa}^2) + 2x_{oa}k\cos(\beta)\cos(\alpha) + 2y_{oa}k\cos(\beta)\sin(\alpha) + 2z_{oa}k\sin(\beta)$$
$$+ 2x_{ob}k\cos(\beta)\cos(\alpha) + 2y_{ob}k\cos(\beta)\sin(\alpha) + 2z_{ob}k\sin(\beta)$$

$$\overbrace{-2(x_{ok}^2 + y_{ok}^2 + z_{ok}^2)}^{\text{simplify}} = -2\left[(k\cos(\beta)\cos(\alpha) + x_{oa})^2 + (k\cos(\beta)\sin(\alpha) + y_{oa})^2 + (k\sin(\beta) + z_{oa})^2\right]$$

$$= -2\left[k^2\cos^2(\beta)\cos^2(\alpha) + 2kx_{oa}\cos(\beta)\cos(\alpha) + x_{oa}^2 + k^2\cos^2(\beta)\sin^2(\alpha) + 2ky_{oa}\cos(\beta)\sin(\alpha) + y_{oa}^2\right.$$
$$\left. + k^2\sin^2(\beta) + 2kz_{oa}\sin(\beta) + z_{oa}^2\right]$$

$$= -2\left[k^2\cos^2(\beta)\cos^2(\alpha) + k^2\cos^2(\beta)\sin^2(\alpha) + k^2\sin^2(\beta) + 2kx_{oa}\cos(\beta)\cos(\alpha) + 2ky_{oa}\cos(\beta)\sin(\alpha)\right.$$
$$\left. + 2kz_{oa}\sin(\beta) + x_{oa}^2 + y_{oa}^2 + z_{oa}^2\right]$$

$$\cos^2(\alpha) + \sin^2(\alpha) = 1 \qquad \cos^2(\beta) + \sin^2(\beta) = 1$$

$$= -2\left[k^2 + 2kx_{oa}\cos(\beta)\cos(\alpha) + 2ky_{oa}\cos(\beta)\sin(\alpha) + 2kz_{oa}\sin(\beta)\right] \overbrace{-2(x_{oa}^2 + y_{oa}^2 + z_{oa}^2)}^{\text{This gets cancelled}}$$

$$'' = \overbrace{-2\left[k^2 + 2kx_{oa}\cos(\beta)\cos(\alpha) + 2ky_{oa}\cos(\beta)\sin(\alpha) + 2kz_{oa}\sin(\beta)\right]}^{\text{from } -2(x_{ok}^2 + y_{ok}^2 + z_{ok})^2} + 2x_{oa}k\cos(\beta)\cos(\alpha)$$
$$+ 2y_{oa}k\cos(\beta)\sin(\alpha) + 2z_{oa}k\sin(\beta) + 2x_{ob}k\cos(\beta)\cos(\alpha) + 2y_{ob}k\cos(\beta)\sin(\alpha) + 2z_{ob}k\sin(\beta)$$

* Note: $-2\times 2kz_{oa}\sin(\beta) + 2z_{oa}k\sin(\beta) = -2kz_{oa}\sin(\beta)$ *

$$'' = -2k^2 - 2kx_{oa}\cos(\beta)\cos(\alpha) - 2ky_{oa}\cos(\beta)\sin(\alpha) - 2kz_{oa}\sin(\beta) + 2kx_{ob}\cos(\beta)\cos(\alpha)$$
$$+ 2ky_{ob}\cos(\beta)\sin(\alpha) + 2kz_{ob}\sin(\beta)$$

$$d^2 - k^2 - L^2 = -2k^2 + 2k\cos(\beta)\cos(\alpha)[x_{ob} - x_{oa}] + 2k\cos(\beta)\sin(\alpha)[y_{ob} - y_{oa}] + 2k\sin(\beta)[z_{ob} - z_{oa}]$$

$$d^2 + k^2 - L^2 = 2k\cos(\beta)\cos(\alpha)(x_{ob} - x_{oa}) + 2k\cos(\beta)\sin(\alpha)(y_{ob} - y_{oa}) + 2k\sin(\beta)(z_{ob} - z_{oa})$$

$$d^2 + k^2 - L^2 = [2k\cos(\alpha)(x_{ob} - x_{oa}) + 2k\sin(\alpha)(y_{ob} - y_{oa})]\cos(\beta) + [2k(z_{ob} - z_{oa})]\sin(\beta)$$

$$M\sin(\beta) + N\cos(\beta) = P \longrightarrow \boxed{P = d^2 + k^2 - L^2}$$

$$M\sin(\beta) + N\cos(\beta) = r\sin(\beta + x)$$

$$\sin(\beta + x) = \sin(\beta)\cos(x) + \sin(x)\cos(\beta)$$

$$M\sin(\beta) + N\cos(\beta) = r\sin(\beta)\cos(x) + r\sin(x)\cos(\beta)$$

$$M = r\cos(x) \qquad N = r\sin(x)$$

$$\cos^2(x) + \sin^2(x) = 1 \longrightarrow M^2 + N^2 = r^2\cos^2(x) + r^2\sin^2(x) \longrightarrow \boxed{r = \sqrt{M^2 + N^2}}$$

$$\frac{r\sin(x)}{r\cos(x)} = \tan(x) = \frac{N}{M} \longrightarrow \boxed{x = \tan^{-1}\left(\frac{N}{M}\right)}$$

$$\therefore M\sin(\beta) + N\cos(\beta) = \sqrt{M^2+N^2}\,\sin\left(\beta + \tan^{-1}\left(\frac{N}{M}\right)\right)$$

$$M\sin(\beta) + N\cos(\beta) = P = d^2 + k^2 - L^2$$

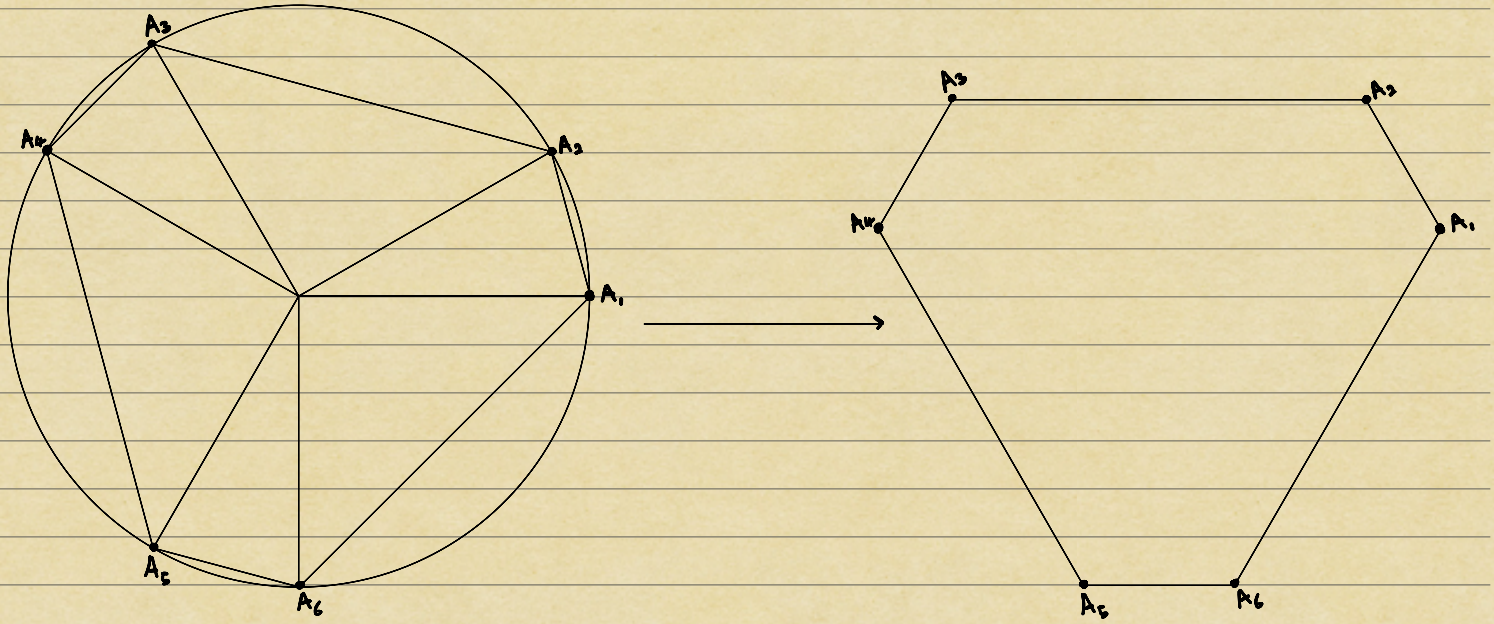
$$\sqrt{M^2+N^2}\,\sin\left(\beta + \tan^{-1}\left(\frac{N}{M}\right)\right) = P \longrightarrow \sin\left(\beta + \tan^{-1}\left(\frac{N}{M}\right)\right) = \frac{P}{\sqrt{M^2+N^2}}$$

$$\boxed{\therefore \beta = \sin^{-1}\left(\frac{P}{\sqrt{M^2+N^2}}\right) - \tan^{-1}\left(\frac{N}{M}\right)}$$

$$\boxed{\begin{aligned} P_i &= d_i^2 + k^2 - L^2 \\ N_i &= 2k(z_{ob,i} - z_{oa,i}) \\ M_i &= 2k\cos(\alpha_i)(x_{ob,i} - x_{oa,i}) + 2k\sin(\alpha_i)(y_{ob,i} - y_{oa,i}) \end{aligned}}$$

Final Geometry Design

Using a 40°-80° arrangement for servos:



Real World Constraints

- The servos placed at the corners of the hexagon is not practical
- To make it more feasible, the servos will be placed in pairs along the longest edges of the hexagon

$\beta_1 = \pi - \beta_1$

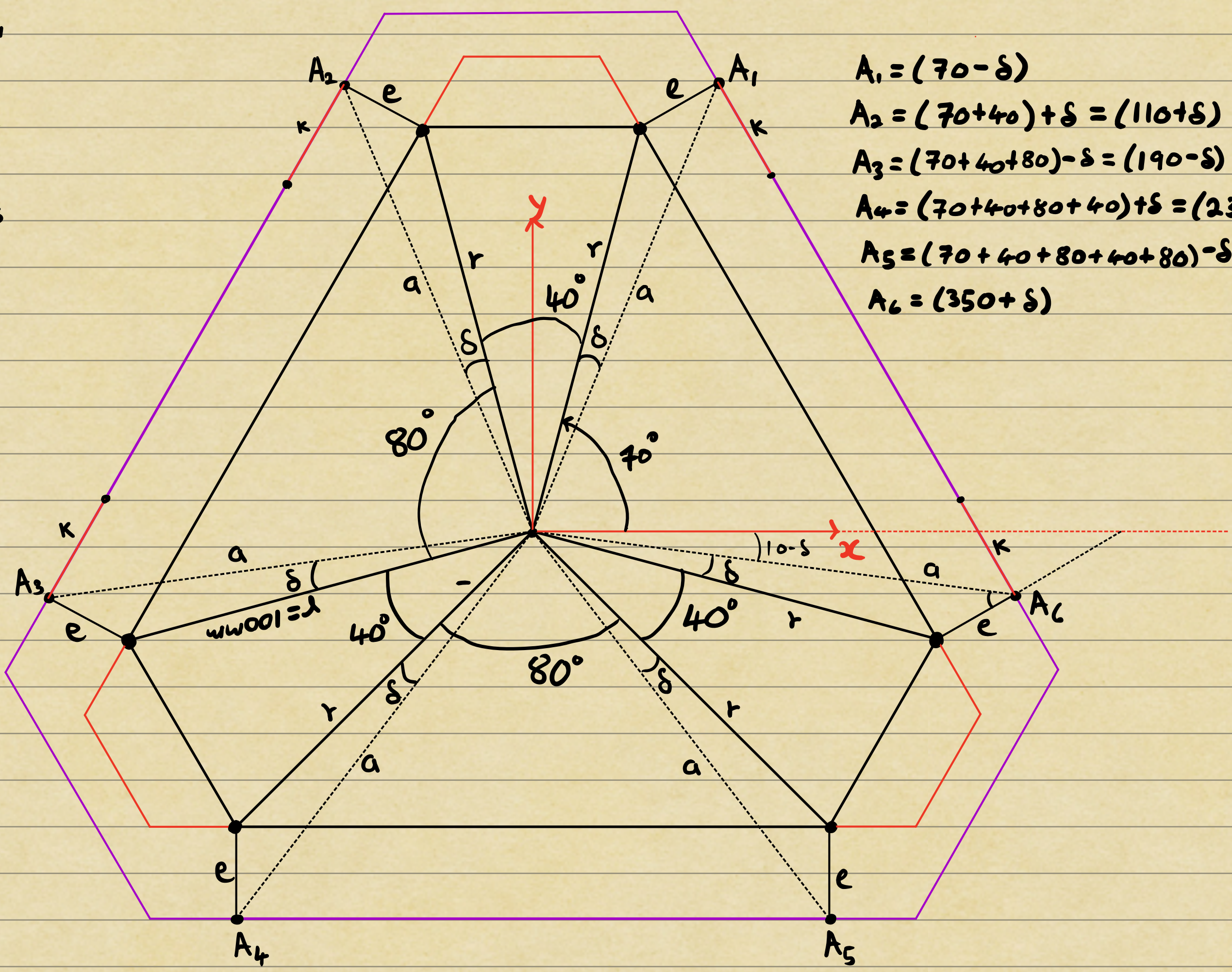
$\beta_2 = \beta_2$

$\beta_3 = \pi - \beta_3$

$\beta_4 = \beta_4$

$\beta_5 = \pi - \beta_3$

$\beta_6 = \beta_6$



$A_1 = (70 - \delta)$

$A_2 = (70+40) + \delta = (110+\delta)$

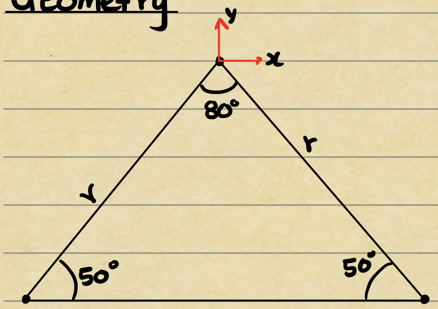
$A_3 = (70+40+80) - \delta = (190-\delta)$

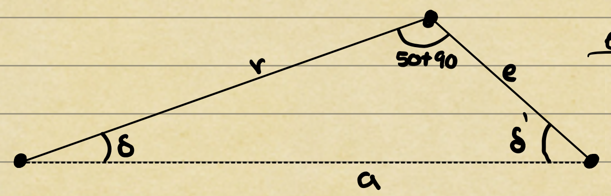
$A_4 = (70+40+80+40) + \delta = (230+\delta)$

$A_5 = (70+40+80+40+80) - \delta = (310-\delta)$

$A_6 = (350+\delta)$

Geometry





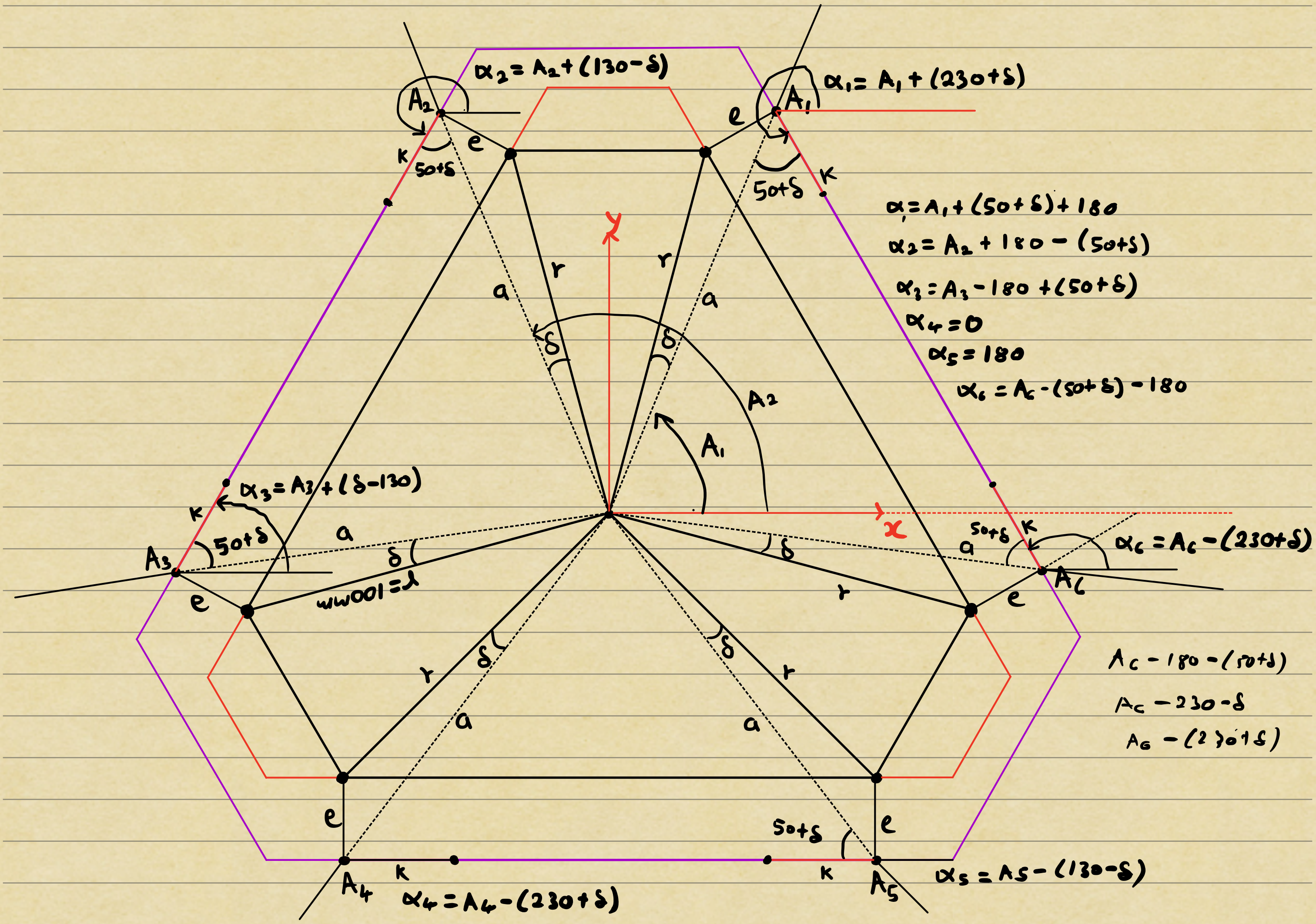
$$e^2 = r^2 + L^2 - 2rL\cos(\delta)$$

$$\frac{e^2 - r^2 - L^2}{-2rL}$$

$$\frac{r^2 + L^2 - e^2}{2rL}$$

$\boxed{a = \sqrt{e^2 + r^2 - 2er\cos(140)}}$ $\boxed{\delta = \cos^{-1}\left(\frac{a^2 + r^2 - e^2}{2ar}\right)}$ → • e is the distance from edge of platfor to servo axle

• r is the radius of the circle



Angles Summary

$A_1 = (70-\delta)^\circ$	$\alpha_1 = A_1 + (230+\delta)$	$\alpha_1 = 300^\circ$
$A_2 = (110+\delta)^\circ$	$\alpha_2 = A_2 + (130-\delta)$	$\alpha_2 = 240^\circ$
$A_3 = (190-\delta)^\circ$	$\alpha_3 = A_3 + (\delta-130)$	$\alpha_3 = 60^\circ$
$A_4 = (230+\delta)^\circ$	$\alpha_4 = A_4 - (230+\delta)$	$\alpha_4 = 0^\circ$
$A_5 = (310-\delta)^\circ$	$\alpha_5 = A_5 - (130-\delta)$	$\alpha_5 = 180^\circ$
$A_6 = (350+\delta)^\circ$	$\alpha_6 = A_6 - (230+\delta)$	$\alpha_6 = 120^\circ$

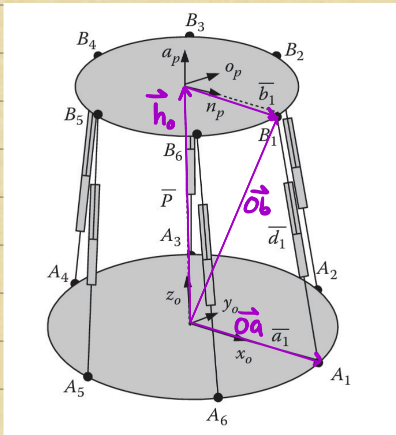
Platform Home Orientation

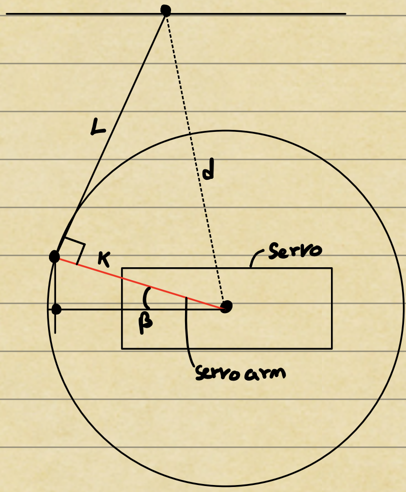
The "Home" orientation for the platform can be defined with 2 conditions:

(1. The push-rod is perpendicular to the servo arm

(2. The platform is Level and there are no rotations about the principal x, y, z axes.

Per the above conditions, the "home" pose for the platform can be mathematically as follows:





(1. $d^2 = L^2 + K^2$

(2. $\theta_x = 0 \quad \theta_y = 0 \quad \theta_z = 0 \longrightarrow \bar{P} = P_z = h_o$

$$|\vec{d}| = -|\vec{oa}| + |\vec{h_o}| + |\vec{b}| \longrightarrow \vec{oa} = \begin{bmatrix} x_{oa,i} \\ y_{oa,i} \\ z_{oa,i} \\ 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} x_{b,i} \\ y_{b,i} \\ z_{b,i} \\ 1 \end{bmatrix} \quad \vec{h_o} = \begin{bmatrix} 0 \\ 0 \\ h_o \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} d_{ix} \\ d_{iy} \\ d_{iz} \\ 1 \end{bmatrix} = \begin{bmatrix} b_{ix} \\ b_{iy} \\ b_{iz} \\ 1 \end{bmatrix} - \begin{bmatrix} a_{ix} \\ a_{iy} \\ a_{iz} \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ h_o \\ 1 \end{bmatrix} \longrightarrow d = \sqrt{d_x^2 + d_y^2 + d_z^2} = \sqrt{(x_{b,i} - x_{oa,i})^2 + (y_{b,i} - y_{oa,i})^2 + (z_{b,i} - z_{oa,i} + h_o)^2}$$

$$d^2 = L^2 + K^2 \longrightarrow L^2 + K^2 = (x_{b,i} - x_{oa,i})^2 + (y_{b,i} - y_{oa,i})^2 + (z_{b,i} - z_{oa,i} + h_o)^2$$

$$(z_{b,i} - z_{oa,i} + h_o)^2 = L^2 + K^2 - (x_{b,i} - x_{oa,i})^2 - (y_{b,i} - y_{oa,i})^2$$

$$\boxed{\therefore h_o = \sqrt{L^2 + K^2 - (x_{b,i} - x_{oa,i})^2 - (y_{b,i} - y_{oa,i})^2} + (z_{oa,i} - z_{b,i})} \text{ — } z_{oa} = 0$$

$$\begin{bmatrix} x_{ob,i} \\ y_{b,i} \\ z_{ob,i} \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + R \begin{bmatrix} x_{b,i} \\ y_{b,i} \\ z_{b,i} \end{bmatrix}$$

$$M\sin(\beta) + N\cos(\beta) = P \longrightarrow \boxed{P = d^2 + K^2 - L^2}$$

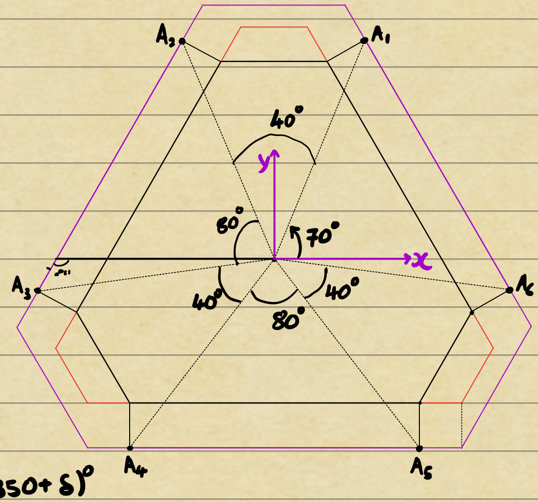
$$z_{ob} = h_o \longrightarrow \boxed{M_{o,i} = 2K(h_o - z_{oa})}$$

$$\boxed{N_{o,i} = 2K\cos(\alpha_i)(x_{ob} - x_{oa,i}) + 2K\sin(\alpha_i)(y_{ob} - y_{oa,i})}$$

$$\boxed{\beta_o = \sin^{-1}\left(\frac{P}{\sqrt{M_{o,i}^2 + N_{o,i}^2}}\right) - \tan^{-1}\left(\frac{N_{o,i}}{M_{o,i}}\right)}$$

Calculation Summary

$$\bar{P}=\begin{bmatrix}P_x\\P_y\\P_z\\1\end{bmatrix} \quad \bar{a}=\begin{bmatrix}x_{oa,i}\\y_{oa,i}\\z_{oa,i}\\1\end{bmatrix} \quad \bar{b}=R_xR_yR_z\begin{bmatrix}x_{b,i}\\y_{b,i}\\z_{b,i}\\1\end{bmatrix}$$



$x_{oa,i}=a\cos(A_i) \rightarrow a=100\,mm$

$A_1=(70-\delta)^\circ,\ A_2=(110+\delta)^\circ,\ A_3=(190-\delta)^\circ,\ A_4=(230+\delta)^\circ,\ A_5=(310-\delta)^\circ,\ A_6=(350+\delta)^\circ$

$L=116.1185\,mm$

$\delta=6.38829^\circ$

$x_{ob,i}=b\sin(B_i) \longrightarrow b=82.156\,mm$

$A_1=63.6117$

$B_1=85.451^\circ$ $B_2=94.549^\circ$ $B_3=205.451^\circ$ $B_4=214.549^\circ$ $B_5=325.451^\circ$ $B_6=334.549^\circ$ $\longrightarrow$ Small edge

$B_1=37.95^\circ$ $B_2=142.05^\circ$ $B_3=157.95^\circ$ $B_4=262.05^\circ$ $B_5=277.95^\circ$ $B_6=22.05^\circ$ $\longrightarrow$ Modified Long edge

$a^2=e^2+r^2-2er\cos(140) \longrightarrow \boxed{\delta=\cos^{-1}\left(\frac{a^2+r^2-e^2}{2ar}\right)} \longrightarrow$ • e is the distance from edge of platfor to servo axle

• r is the radius of the circle

$e=22.55\,mm \quad r=100\,mm$

$k=24\,mm$

$$\begin{bmatrix}d_{ix}\\d_{iy}\\d_{iz}\\1\end{bmatrix}=\begin{bmatrix}P_x\\P_y\\P_z\\1\end{bmatrix}+R_xR_yR_z\begin{bmatrix}x_{b,i}\\y_{b,i}\\z_{b,i}\\1\end{bmatrix}-\begin{bmatrix}x_{oa,i}\\y_{oa,i}\\z_{oa,i}\\1\end{bmatrix} \longrightarrow \boxed{d_i=\sqrt{d_{x,i}^2+d_{y,i}^2+d_{z,i}^2}}$$

Leg Length

Servo Angle:

$\boxed{P=d^2+k^2-L^2}$ $\boxed{N_i=2k\cos(\alpha_i)(x_{ob}-x_{oa,i})+2k\sin(\alpha_i)(y_{ob}-y_{oa,i})}$ $\boxed{M_i=2k(z_{ob}-z_{oa,i})}$

$$\boxed{\therefore\ \beta=\sin^{-1}\left(\frac{P}{\sqrt{M_i^2+N_i^2}}\right)-\tan^{-1}\left(\frac{N_i}{M_i}\right)} \longrightarrow$$

• d: Leg Length
• k: Servo arm length
• L: Connecting rod length

$\alpha_1=300^\circ$
$\alpha_2=240^\circ$
$\alpha_3=60^\circ$
$\alpha_4=0^\circ$
$\alpha_5=180^\circ$
$\alpha_6=120^\circ$

• **Home Pose:**

$\boxed{P_x=0,\ P_y=0,\ P_z=h_o,\ \theta_x=0,\ \theta_y=0,\ \theta_z=0}$ $\boxed{\therefore\ h_o=\sqrt{L^2+k^2-(x_{b,i}-x_{oa,i})^2-(y_{b,i}-y_{oa,i})^2}+(z_{oa,i}-z_{b,i})}$

Platform home height

• **Home Servo Angle:**

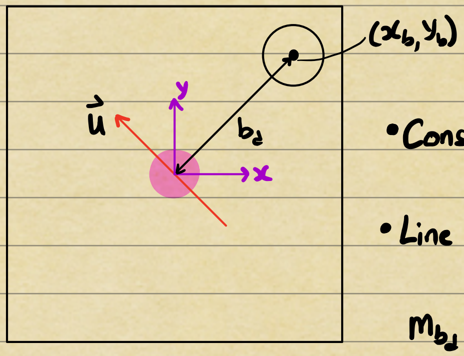
$\boxed{M_{o,i}=2k(h_o-z_{oa})}$ $\boxed{N_{o,i}=2k\cos(\alpha_i)(x_{ob}-x_{oa,i})+2k\sin(\alpha_i)(y_{ob}-y_{oa,i})}$ $\boxed{\alpha_i=\mu_i-90}$

$$\boxed{\beta_o=\sin^{-1}\left(\frac{P}{\sqrt{M_{o,i}^2+N_{o,i}^2}}\right)-\tan^{-1}\left(\frac{N_{o,i}}{M_{o,i}}\right)}$$

$$\boxed{d_o=\sqrt{(x_{b,i}-x_{oa,i})^2+(y_{b,i}-y_{oa,i})^2+(z_{b,i}-z_{oa,i}+h_o)^2}}$$

Ball Balancing

In order to get the ball to the centre of the platform, the platform must be rotated about an axis perpendicular to a line from the origin to the ball. Given that the x,y position of the ball is known, a unit vector representing the required rotation axis can be derived mathematically.



$$\boxed{b_d = \sqrt{(x_b-0)^2+(y_b-0)^2}}$$

• Consider line b_d: $m_{b_d} = \frac{(y_b-0)}{(x_b-0)} = \frac{y_b}{x_b}$ ← Gradient

• Line b_d is perpendicular to the required axis of rotation:

gradient of required axis

$$m_{b_d} \times m_u = -1 \longrightarrow m_u = -\frac{1}{m_{b_d}} = -\frac{x_b}{y_b}$$

∴ Euation of the axis: $y - y_1 = m_a(x - x_1) \longrightarrow \boxed{y = -\frac{x_b}{y_b}x}$

• The desired axis of rotation is a unit vector:

$$\vec{u} = u_x\hat{i} + u_y\hat{j} \longrightarrow |\vec{u}| = \sqrt{u_x^2 + u_y^2} = 1$$

$$\tan(\theta) = \frac{\Delta y}{\Delta x} \longrightarrow \tan(\theta) = m \longrightarrow \frac{u_y}{u_x} = \tan(\theta) = m_u \qquad a^2 + a^2 m_u^2$$

$$\frac{u_y}{u_x} = -\frac{x_b}{y_b} \longrightarrow u_y = -\frac{u_x x_b}{y_b} = m_u u_x$$

$$\sqrt{u_x^2 + u_x^2 m_u^2} = 1 \longrightarrow u_x^2(1+m_u^2) = 1^2 \longrightarrow u_x^2 = \frac{1}{(1+m_u^2)} \longrightarrow \boxed{u_x = \sqrt{\frac{1}{\left(1+\frac{x_b^2}{y_b^2}\right)}}} \qquad \boxed{u_y = -\frac{u_x x_b}{y_b}}$$

$$\boxed{\vec{u} = \pm\sqrt{\frac{1}{\left(1+\frac{x_b^2}{y_b^2}\right)}}\left(\hat{i} - \frac{x_b}{y_b}\hat{j}\right)}$$

→ The sign of the axis unit vector components can be chosen based on the quadrant the ball is in.

Quaternion Representation

Quaternions are 4-dimensional mathematical objects that can be used to represent rotations. They are closely related to the axis-angle representation. The structure of a quaternion is as follows:

$$\boxed{q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}}$$

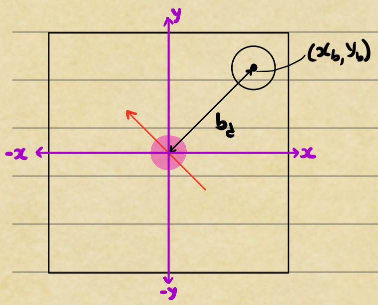
A rotation quaternion can be obtained from the axis-angle representation:

$$\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$$

$$q_1 = u_x\sin\left(\frac{\theta_u}{2}\right) \quad q_2 = u_y\sin\left(\frac{\theta_u}{2}\right) \quad q_3 = u_z\sin\left(\frac{\theta_u}{2}\right) \quad q_0 = \cos\left(\frac{\theta_u}{2}\right) \longrightarrow \frac{\theta_u}{2} = \theta$$

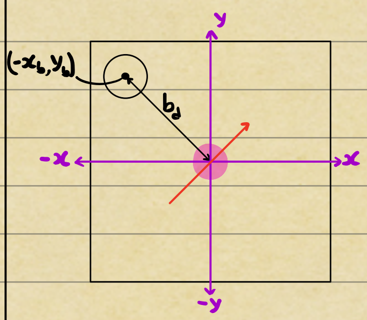
$$\therefore \boxed{q = u_x\sin(\theta)\hat{i} + u_y\sin(\theta)\hat{j} + u_z\sin(\theta)\hat{k} + \cos(\theta)}$$

Case 1



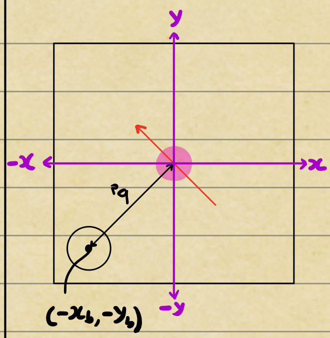
$-u_x \quad +\theta$

Case 2



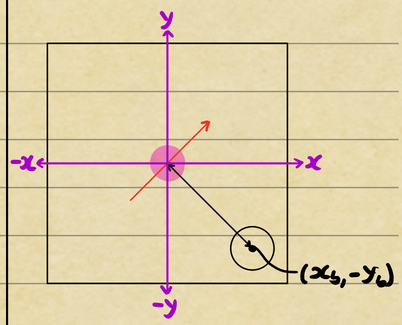
$+u_x \quad -\theta$

Case 3



$-u_x \quad -\theta$

Case 4



$+u_x \quad +\theta$

Converting quaternion to an equivalent Rotation Matrix

$$\boxed{R = \begin{bmatrix} (q_0^2 + q_1^2 - q_2^2 - q_3^2) & (2q_1q_2 - 2q_0q_3) & (2q_1q_3 + 2q_0q_2) \\ (2q_1q_2 + 2q_0q_3) & (q_0^2 - q_1^2 + q_2^2 - q_3^2) & (2q_2q_3 - 2q_0q_1) \\ (2q_1q_3 - 2q_0q_2) & (2q_2q_3 + 2q_0q_1) & (q_0^2 - q_1^2 - q_2^2 + q_3^2) \end{bmatrix}}$$

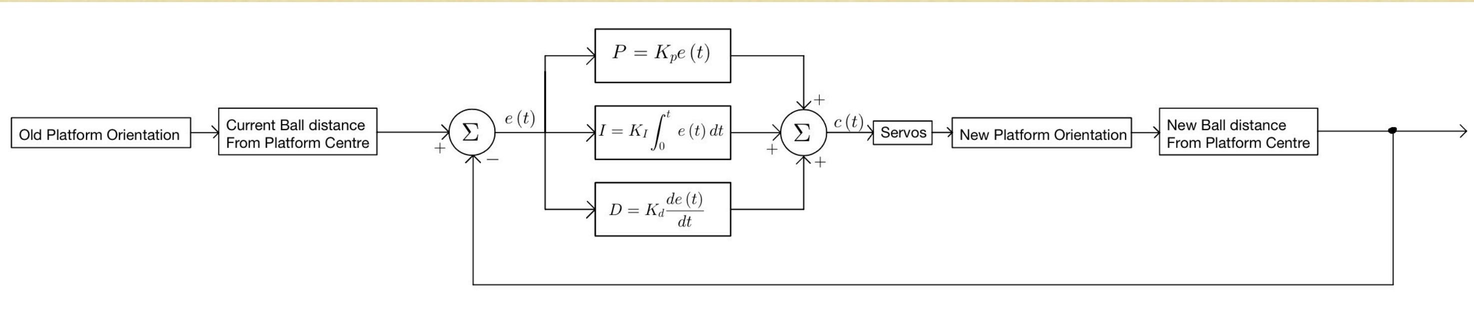
Platform Control System

- PID control will be utilized to keep the ball at the centre of the platform

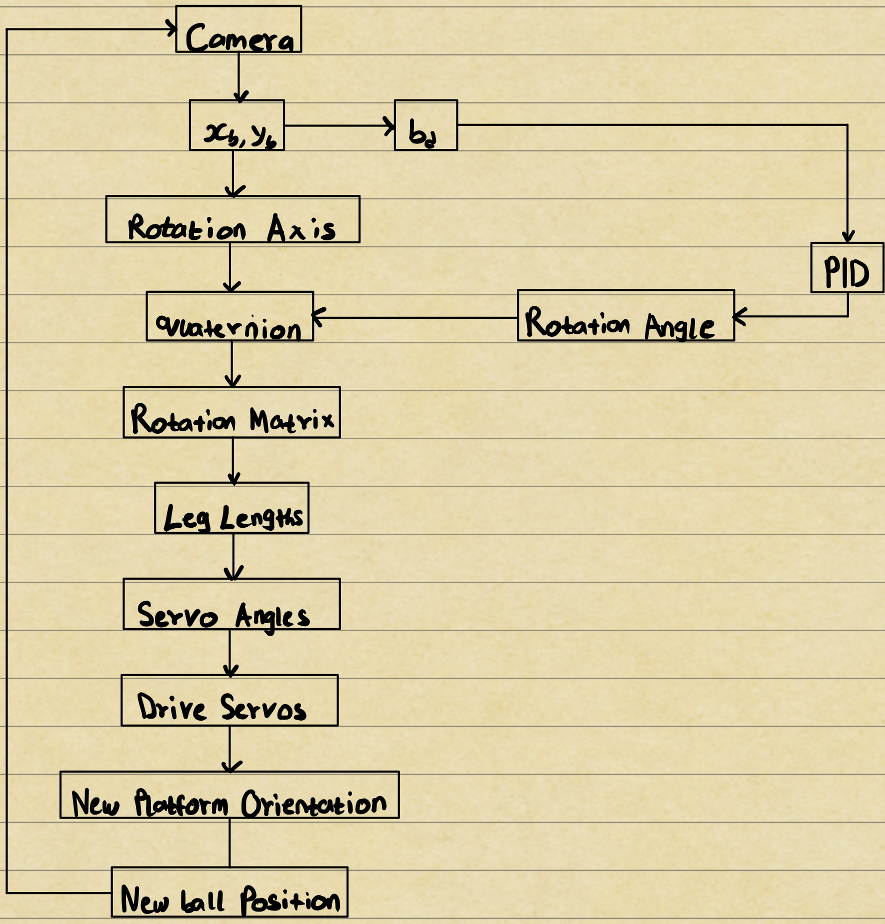
- There are 2 approaches that could be used to design the control system:

(1) Quaternion based PID: - Computes 1 axis of rotation required for the ball to get to the centre
- Requires the Euclidian distance of the centre of the ball to the platform centre as the control input.

(2) Rotation matrix based PID: - Requires 2 separate PID's to control the ball's x and y position
- Uses a Rotation about the x-axis and rotation about the y-axis to control the platform.



Algorithm Structure



Computer Vision

Object detection is comprised of 3 processes:

(1. Object is acknowledged/detected

(2. Object is classified (object type)

(3. Object is localized (bounding box)

Generally, modern object detection algorithms use Convoluted Neural Networks to process images. The network needs to be trained with input data.

Training

The network is fed with images that have:

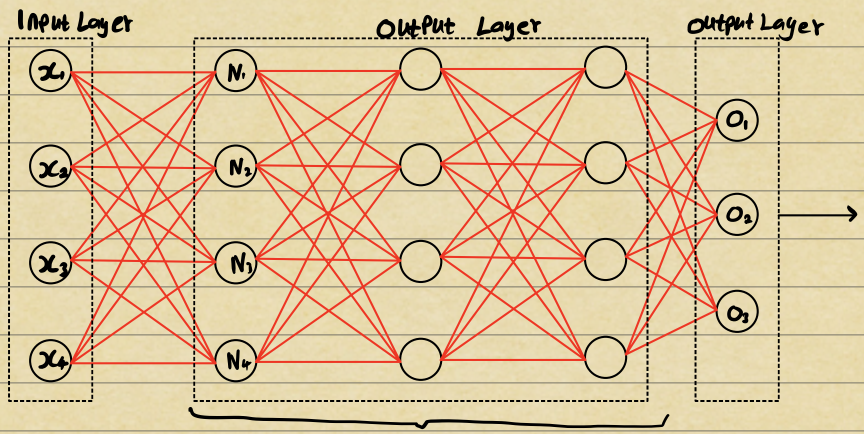
(1. the desired objects with manually drawn bounding boxes

(2. Object classes specified.

The CNN takes in inputs of images and outputs a vector in the following form:

$$\begin{bmatrix} P_c \\ x_c \\ y_c \\ B_w \\ B_h \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \text{Probability of the class} \\ x\text{-Coordinate of bounding box centre} \\ y\text{-Coordinate of bounding box centre} \\ \text{Bounding box width} \\ \text{Bounding box height} \\ \text{Object class 1} \\ \text{Object class 2} \end{bmatrix}$$

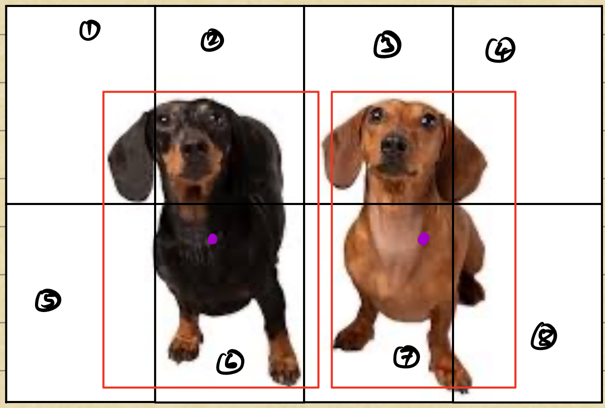
Hence, the training images with bounding boxes must be converted to the above format.



Hidden layer is where the feature extraction occurs

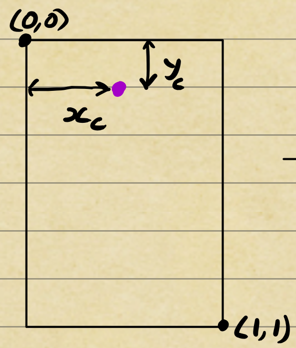
YOLO (You Only Look Once) Algorithm

- Divides the image into grid segments
- A vector is defined for each grid cell
- Only grid cells that contain bounding box centres have a vector with a class probability greater than zero.



e.g cell 1: $\begin{bmatrix} 0 \\ - \\ - \\ - \\ - \\ - \\ - \end{bmatrix}$

Cell 6 :



$$\begin{bmatrix} P_c \\ x_c \\ y_c \\ B_w \\ B_h \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ x_c \\ y_c \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Class 2